ON THE STABILITY OF MOTION OF A HEAVY SOLID ABOUT A FIXED POINT IN A CERTAIN SPECIAL CASE

(OB USTOICHIVOSTI DVIZHENIIA TIAZHELOGO TELA VOKRUG NEPODVIZHNOI TOCHKI V ODNOM CHASTNOM SLUCHAE)

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Iu.A. ARKHANGELSKII
(Moscow)

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1. It is known that the equations of motion of a heavy solid about a fixed point inside, with an arbitrary ellipsoid of inertia about its fixed point, and with its center of gravity in the principal plane of inertia xy, are

$$\begin{aligned} &A\frac{dp}{dt}+(C-B)qr=Mgy_0\gamma'', &&\frac{d\gamma}{dt}=r\gamma'-q\gamma'' \\ &B\frac{dq}{dt}+(A-C)pr=-Mgx_0\gamma'', &&\frac{d\gamma'}{dt}=p\gamma''-r\gamma \\ &C\frac{dr}{dt}+(B-A)pq=Mg(x_0\gamma'-y_0\gamma). &&\frac{d\gamma''}{dt}=q\gamma-p\gamma' \end{aligned} \qquad (1.1)$$

In order to satisfy the conditions for a pendular motion

$$p=q=\gamma''=0 \qquad (1.2)$$

the quantities r, γ, γ' must satisfy the equations

$$\frac{dr}{dt}=\frac{Mg}{C}(x_0\gamma'-y_0\gamma), \qquad \frac{d\gamma}{dt}=r\gamma', \qquad \frac{d\gamma'}{dt}=-r\gamma \qquad (1.3)$$

which follow from the system (1.1).

The substitution $r=\dot\phi$, $\gamma=\sin\phi$, $\gamma'=\cos\phi$, reduces the equations (1.3) to a single second-order equation

$$\frac{d^2\varphi}{dt^2}-R^2\cos(\varphi+\varphi_0)=0 \qquad \left(R^2=\frac{Mg}{C}\sqrt{x_0^2+y_0^2}, \quad \operatorname{tg}\varphi_0=\frac{y_0}{x_0}\right) \quad (1.4)$$

The solution of this equation can be expressed through Jacobi elliptic functions [1] with the period $\omega=4K/R$; then the quantities r, γ, γ' are expressed by

$$r_1 = 2Rk\,\mathrm{cn}\,(Rt) \qquad (k = \sin{}^1/_2\,\psi_0) \tag{1.5}$$
$$\gamma_1 = \cos\varphi_0 + 2k\sin\varphi_0\,\mathrm{sn}\,(Rt)\,\mathrm{dn}\,(Rt) - 2k^2\cos\varphi_0\,\mathrm{sn}^2\,(Rt)$$
$$\gamma_1' = \sin\varphi_0 - 2k\cos\varphi_0\,\mathrm{sn}\,(Rt)\,\mathrm{dn}\,(Rt) - 2k^2\sin\varphi_0\,\mathrm{sn}^2\,(Rt)$$

Here k is the modulus of the corresponding elliptic integral, ψ_0 is the greatest displacement angle of the center of gravity of the solid from the position of stable equilibrium, and the quantity K entering in the expression for the period of these functions is the full elliptic integral of the first kind which corresponds to the given k.

We shall consider stability in the first approximation of the motion as defined by (1.2) and (1.5), assuming that the parameter k is sufficiently small, and begin the analysis with the case $x_0 < y_0$.

2. Denoting the variations of the variables in the perturbed motion by ξ, η, ζ, u, v, w, we obtain

$$p = \xi, \quad q = \eta, \quad r = r_1 + u, \quad \gamma = \gamma_1 + v, \quad \gamma' = \gamma_1' + w, \quad \gamma'' = \zeta \tag{2.1}$$

and the equations in Poincare's variations are

$$\frac{d\xi}{dt} = a_0 r_1 \eta + y_0'\zeta, \qquad \frac{d\eta}{dt} = -b_0 r_1 \xi - x_0'\zeta, \qquad \frac{d\zeta}{dt} = \gamma_1\eta - \gamma_1'\xi \tag{2.2}$$
$$\frac{du}{dt} = x_0''w - y_0''v, \qquad \frac{dv}{dt} = r_1 w + \gamma_1' u, \qquad \frac{dw}{dt} = -r_1 v - \gamma_1 u \tag{2.3}$$

where

$$\frac{B-C}{A} = a_0, \qquad \frac{A-C}{B} = b_0$$
$$\frac{Mgx_0}{B} = x_0', \qquad \frac{Mgy_0}{A} = y_0', \qquad \frac{Mgx_0}{C} = x_0'', \qquad \frac{Mgy_0}{C} = y_0'' \tag{2.4}$$

In this way the sixth-order system of equations for the perturbed motion in the first approximation is broken down into two independent linear systems of the third order with periodic coefficients; further, the product of the roots of the characteristic equation of each independent system equals unity.

Utilizing the first three integrals of the system (1.1)

$$\gamma^2 + \gamma'^2 + \gamma''^2 = 1, \qquad Ap\gamma + Bq\gamma' + Cr\gamma'' = C_1 \tag{2.5}$$
$$Ap^2 + Bq^2 + Cr^2 - 2Mg\,(x_0\gamma + y_0\gamma') = C_2$$

and the formulas (2.1), we obtain the following first integrals of the system (2.2) and (2.3):

$$\gamma_1 v + \gamma_1' w = 0 \tag{2.6}$$

$$A\xi\gamma_1 + B\eta\gamma_1' + C\zeta r_1 = H \tag{2.7}$$

$$r_1 u - x_0'' v - y_0'' w = G \tag{2.8}$$

where H and G are arbitrary constants. The system (2.2) has one first integral (2.7), and the system (2.3) has two first integrals (2.6) and (2.7).

We shall examine the system (2.3). Taking into account that (2.3) was derived from the autonomous system (1.1) there exists, on the strength of Poincaré's theorem, not only the first two integrals but also the periodic solution

$$u_1 = \dot{r}_1, \qquad v_1 = \dot{\gamma}_1, \qquad w_1 = \dot{\gamma}_1' \tag{2.9}$$

We shall show that the second solution u_2, v_2, w_2. of the system (2.3) is not periodic. Indeed, using the integral (2.6) to reduce the system (2.3) to a single second-order equation, with respect to u for example, we shall find by the Liouville formula the particular solution

$$u_2 = \dot{r}_1 \int L(t)\,dt \qquad \left(L(t) = \frac{1 - 2k^2 \operatorname{sn}^2(Rt)}{4k^2R^4 \operatorname{sn}^2(Rt) \operatorname{dn}^2(Rt)}\right) \tag{2.10}$$

The function $L(t)$ is periodic with the period $\omega_1 = 1/2\ \omega$ and is discontinuous at the points where t is a multiple of the period ω. The function u_2, however, is a continuous function in any finite interval of time, and as the mean value of $L(t)$ is non-zero, the function u_2 increases with time without bounds. The same can be said about the functions v_2 and w_2; hence to the triple unit root of the characteristic equation there corresponds not more than two groups of solutions.

Thus, for the trivial solution of (2.3) the conditions of stability with respect to the variables u, v, w, are not satisfied if the initial perturbations are not restricted in any way, and the motion of a solid determined by (1.2) and (1.5) will be unstable in the first approximation.

In the general solution of (2.3)

$$u = C_1 u_1 + C_2 u_2, \qquad v = C_1 v_1 + C_2 v_2, \qquad w = C_1 w_1 + C_2 w_2$$

(the existence of the integral (2.6) makes the third arbitrary constant equal identically zero) the initial perturbations will be subjected to the condition $C_2 = 0$.

Taking the initial instant of the time $t = 0$, we obtain from the formulas (2.9) and (2.10) the conditions for the initial perturbations in the form $u(0) = 0$.

It should be mentioned that the above condition could be also obtained from the integral (2.8) by setting $G = 0$.

From the above considerations it follows that the characteristic equation of the system (2.3) has a triple unit root and if the initial perturbations are subject to the condition $u(0) = 0$, then the solution of the system (2.3) is periodic

$$u = C_1\dot{r}_1, \qquad v = C_1\dot{\gamma}_1, \qquad w = C_1\dot{\gamma}_1' \tag{2.11}$$

(here C_1 is an arbitrary constant) and the conditions of stability for this system will be satisfied.

3. Let us consider now the system (2.2). Utilizing the first integral (2.7) we can transform this system to a system of two equations of the first order

$$\frac{d\xi}{d\tau} = f_{11}\xi + f_{12}\zeta + F_1 H, \qquad \frac{d\zeta}{d\tau} = f_{21}\xi + f_{22}\zeta + F_2 H \tag{3.1}$$

Here

$$f_{11} = -l(1-b)\frac{\gamma_1 r_1}{\gamma_1'}, \qquad f_{12} = -l\left[a(1-b)\frac{r_1^2}{\gamma_1'} - y_0'\right]$$

$$f_{21} = -\frac{l}{a}\left[b\frac{1}{\gamma_1'} + (a-b)\gamma_1'\right], \qquad f_{22} = -lb\frac{r_1\gamma_1}{\gamma_1'}$$

$$F_1 = \frac{la(1-b)}{C}\frac{r_1}{\gamma_1'}, \qquad F_2 = \frac{l}{C}b\frac{\gamma_1}{\gamma_1'} \tag{3.2}$$

$$l = \frac{\omega}{\pi}, \qquad t = l\tau, \qquad a = \frac{C}{A}, \qquad b = \frac{C}{B}$$

Equations (3.1) reduced to a single equation of the second order are as follows:

$$\frac{d^2\sigma}{d\tau^2} + F(\tau)\sigma = N(\tau)H \tag{3.3}$$

Here

$$\xi = \sqrt{f_{12}\gamma_1'\sigma} \tag{3.4}$$

$$F(\tau) = P - \frac{1}{4}Q^2 - \frac{1}{2}\frac{dQ}{d\tau}, \qquad N(\tau) = \sqrt{\frac{f_{12}}{\gamma_1'}}\left(\frac{d}{d\tau}\frac{F_1}{f_{12}} - f_{22}\frac{F_1}{f_{12}} + F_2\right)$$

$$Q = -\left(\frac{1}{f_{12}}\frac{df_{12}}{d\tau} + f_{11} + f_{22}\right), \qquad P = -\left(f_{12}\frac{d}{d\tau}\frac{f_{11}}{f_{12}} + f_{21}f_{12} - f_{11}f_{22}\right)$$

The coefficient and the free member of Equation (3.3) are continuous periodic functions of the time with the period π, expansible in Fourier series, and also for sufficiently small values of k are analytic functions of k.

We shall find the expansion of the coefficient $F(\tau)$ in powers of the parameter k, truncating all terms of degree higher than k^3.

Denoting

$$\nu = \frac{Mg}{C}, \quad \mu = 1 + s^2, \quad s = \frac{x_0}{y_0}, \quad \lambda^2 = 4\frac{b(\mu - 1) + a}{\mu} \tag{3.5}$$

we obtain

$$l = \frac{2}{\nu\sqrt[4]{x_0^2 + y_0^2}}\left(1 + \frac{k^2}{4}\right)$$

$$f_{11} = -4(1 - b)[ke_0 + k^2e_1 + k^3e_2], \qquad f_{12} = \frac{2\nu a y_0}{\sqrt[4]{x_0^2 + y_0^2}}[1 + k^2e_3 + k^3e_4]$$

$$f_{21} = -\frac{2\sqrt[4]{x_0^2 + y_0^2}}{\nu a y_0}\left[\frac{\lambda^2}{4} + ke_5 + k^2e_6 + k^3e_7\right], \qquad f_{22} = -4b[ke_0 + k^2e_1 + k^3e_2]$$

hence the coefficient in the equation (3.3) will be

$$F(\tau;\ \lambda^2;\ k) = \lambda^2 + k\varkappa_1(\tau;\ \lambda^2;\ k) + k^2\varkappa_2(\tau;\ \lambda^2;\ k) + k^3\varkappa_3(\tau;\ \lambda^2;\ k) + \ldots$$

Here

$$\varkappa_1 = 2(3 - 2b)\frac{de_0}{d\tau} + 4e_5$$

$$\varkappa_2 = 2(3 - 2b)\frac{de_1}{d\tau} + 4e_6 + \lambda^2e_3 + 4e_0^2[4b(1 - b) - 1] - \frac{1}{2}\frac{d^2e_3}{d\tau^2}$$

$$\varkappa_3 = 2(3 - 2b)\frac{de_2}{d\tau} + 4e_7 + \lambda^2e_4 + 8e_1e_0[4b(1 - b) - 1] - \\ - \frac{1}{2}\frac{d^2e_4}{d\tau^2} - 2(1 - 2b)e_0\frac{de_3}{d\tau} + 4e_3e_5$$

$$e_0 = s\,\mathrm{cn}(lR\tau), \quad e_1 = 2\mu\,\mathrm{sn}(lR\tau)\,\mathrm{cn}(lR\tau), \quad e_2 = s\,\mathrm{cn}(lR\tau)\left[\tfrac{1}{4} + 4\mu\,\mathrm{sn}^2(lR\tau)\right]$$

$$e_3 = \tfrac{1}{4} - 4(1 - b)\mu\,\mathrm{cn}^2(lR\tau), \qquad e_4 = -8(1 - b)s\mu\,\mathrm{sn}(lR\tau)\,\mathrm{cn}^2(lR\tau)$$

$$e_5 = 2s\,\mathrm{sn}(lR\tau)\left(2b - \tfrac{1}{4}\lambda^2\right), \qquad e_6 = \tfrac{1}{16}\lambda^2 + 2\,\mathrm{sn}^2(lR\tau)\left(2b\mu - \tfrac{1}{4}\lambda^2\right)$$

$$e_7 = s\,\mathrm{sn}(lR\tau)\left[\tfrac{1}{8}(8b - \lambda^2) + 8b\mu\,\mathrm{sn}^2(lR\tau) - \mathrm{sn}(lR\tau)\left(2b - \tfrac{1}{4}\lambda^2\right)\right]$$

In the above formulas the quantity a is replaced by the corresponding expression from Formula (3.5).

All the quantities $\kappa_1(\tau;\ \lambda^2;\ k)$ $(i = 1, 2, \ldots)$ are polynomials in b, s, λ^2 and with respect to the parameters k and λ^2 the coefficient $F(\tau;\ \lambda^2;\ k)$ could be expressed in the form

$$F(\tau;\ \lambda^2;\ k) = \lambda^2[1 + kf_1(\tau;\ k)] + kf_2(\tau;\ k) \tag{3.6}$$

where the functions f_1 and f_2 do not depend on λ^2.

From the Fourier series expansion of the Theta function [1] we obtain the following relation

$$\operatorname{sn}(lR\tau) = \sin 2\tau - \tfrac{1}{16}k^2(\sin 2\tau - \sin 6\tau)$$
$$\operatorname{cn}(lR\tau) = \cos 2\tau - \tfrac{1}{16}k^2(3\cos 2\tau - \cos 6\tau)$$

Using the above relations we could write

$$\varkappa_{01} = \rho_1(\lambda^2)\sin 2\tau, \qquad \varkappa_{02} = \rho_2(\lambda^2) + \rho_3(\lambda^2)\cos 4\tau$$
$$\varkappa_{03} = \rho_4(\lambda^2) + \rho_5(\lambda^2)\sin 2\tau + \rho_6(\lambda^2)\sin 4\tau + \rho_7(\lambda^2)\sin 6\tau \tag{3.7}$$

Here

$$\rho_1(\lambda^2) = -2s(\lambda^2 - 12b + 6)$$
$$\rho_2(\lambda^2) = -\left\{\lambda^2\left[\tfrac{1}{2} + 2\mu(1-b)\right] - 8\mu b - 8b(1-b)\mu + 2\mu - 2 + 8b(1-b)\right\}$$
$$\rho_3(\lambda^2) = -\left\{\lambda^2[2(1-b)\mu - 1] - 8\mu(1-b) - 8b\mu(1-b) + 2\mu - 2 + 8b(1-b)\right\} \tag{3.8}$$
$$\rho_4(\lambda^2) = s\left[\tfrac{1}{2}\lambda^2 - 4b\right]$$
$$\rho_5(\lambda^2) = s\left\{2\mu[\lambda^2(1-b) + 22b - 14] - \left[\tfrac{23}{8}\lambda^2 - \tfrac{47}{2}b + \tfrac{3}{4}\right]\right\}$$

The coefficient $F(\tau;\ \lambda^2;\ k)$ assumed the final form

$$F(\tau;\ \lambda^2;\ k) = \lambda^2 + k\varkappa_{01}(\tau;\ \lambda^2) + k^2\varkappa_{02}(\tau;\ \lambda^2) + k^3\varkappa_{03}(\tau;\ \lambda^2) + \ldots \tag{3.9}$$

where κ_{01}, κ_{02}, κ_{03} ... do not depend on the parameter k. We shall consider the homogeneous equation, corresponding to Equation (3.3):

$$\frac{d^2\sigma}{d\tau^2} + F(\tau;\ \lambda^2;\ k)\sigma = 0 \tag{3.10}$$

which determines the stability of the trivial solution of the nonhomogeneous equation.

With regard to this equation it is known [2] that in the neighborhood of every integer there exist two values of λ^*, which are analytic with respect to k^χ ($\chi = 1/2$ or 1), for which the corresponding solution of Equation (3.10) will be periodic with the period π or 2π.

In this case the coefficient $F(\tau;\ \lambda^2;\ k)$ is an analytic function of k^χ, hence the initial conditions must also be analytic functions of k^χ, and the considered periodic solution

$$\sigma^* = \sigma_0^*(\tau) + k^\chi \sigma_1^*(\tau) + \ldots$$

will be an analytic function of k^χ too.

From the equation which determines $\sigma_0^*(\tau)$

$$\frac{d^2\sigma_0^*}{d\tau^2} + n^2\sigma_0^* = 0$$

where n is a nonzero integer, and all λ^* are real and analytic functions of $k(\chi = 1)$, it follows that $\sigma_0^*(\tau)$ cannot be a constant.

Indeed, utilizing the method pointed out in Malkin's book [2] we could obtain on the strength of Formula (3.6) that λ^* satisfies the relation

$$\lambda^{*2} = \int\limits_0^{\beta\pi} \left[\frac{d\bar{\sigma}^*}{d\tau}\frac{d\sigma^*}{d\tau} - kf_2(\tau;\ k)\,\bar{\sigma}^*\ \sigma^*\right] d\tau \left[\int\limits_0^{\beta\pi} [1 + kf_1(\tau;\ k)]\,\bar{\sigma}^*\ \sigma^*\ d\tau\right]^{-1} \quad (\beta = 1,2)$$

($\bar{\sigma}^*$ is a periodic solution, conjugate to σ^*), whose right member is always positive when k is sufficiently small.

We shall show now that when the moments of inertia are related as follows

$$A \geqslant C, \qquad B \geqslant C \tag{3.11}$$

then for Equation (3.10) there exist regions of instability which do not degenerate into a point when k takes on nonzero values.

To determine the boundaries of these regions of instability we shall utilize the well-known method [2] which, as shown above, could be fully applied in our case.

Since the inequalities (3.11) are equivalent to the inequalities $a \leqslant 1$ and $b \leqslant 1$, it follows from Formula (3.5) that $\lambda \leqslant 2$ and there exist only two regions of instability corresponding to $n = 1$ and $n = 2$.

In order to find the first region of instability we substitute in Equation (3.10)

$$\lambda^2 = 1 + \alpha_1 k + \alpha_2 k^2 + \alpha_3 k^3 + \ldots$$

and try to meet this condition by a series solution

$$\sigma = A_0 \cos\tau + B_0 \sin\tau + k\sigma_1 + k^2\sigma_2 + k^3\sigma_3 + \ldots$$

with periodic coefficients, where A_0 and B_0 are arbitrary constants. Then, the equation which determines C_1 is

$$\frac{d^2\sigma_1}{d\tau^2} + \sigma_1 = -\left[\alpha_1 A_0 + \tfrac{1}{2}\rho_1(1)\, B_0\right]\cos\tau - \left[\alpha_1 B_0 + \tfrac{1}{2}\rho_1(1)\, A_0\right]\sin\tau + \ldots \tag{3.12}$$

from which it follows that the necessary condition for periodicity must have the form

$$\alpha_1^{(1,2)} = \pm \tfrac{1}{2}\rho_1(1)$$

When $\rho_1(1) \neq 0$ there exist two distinct solutions for α_1 which generate series determining the boundaries of the regions of instability

$$1 - \tfrac{1}{2}|\rho_1(1)|k + \ldots \leqslant \lambda^2 \leqslant 1 + \tfrac{1}{2}|\rho_1(1)|k + \ldots \tag{3.13}$$

Since $\rho_1(1) = 24s(b - 7/12)$, in order to determine the boundaries of the regions of instability when $b = 7/12$ (the case when $s = 0$ will be examined separately), it is necessary to consider an approximation which follows. In this case the solution of (3.12) will be

$$\sigma_1 = A_1 \cos\tau + B_1 \sin\tau$$

where A_1 and B_1 are arbitrary constants and σ_2 is determined from the equation

$$\frac{d^2\sigma_2}{d\tau^2} + \sigma_2 = -A_0[\alpha_2 + \rho_2(1)]\cos\tau - B_0[\alpha_2 + \rho_2(1)]\sin\tau + \ldots \tag{3.14}$$

The necessary condition for the periodicity of the second approximation will take the form $[\alpha_2 + \rho_2(1)]^2 = 0$ and the general solution of Equation (3.14) will be

$$\sigma_2 = A_2 \cos\tau + B_2 \sin\tau + \sigma_2^*$$

where A_2 and B_2 are arbitrary constants and σ_2^* is a particular solution of this equation. From the equation for the third approximation

$$\frac{d^2\sigma_3}{d\tau^2} + \sigma_3 = -\left\{A_0[\alpha_3 + \rho_4(1)] + \tfrac{1}{2}\left[\rho_5(1) - \rho_2(1)\frac{\partial\rho_1}{\partial(\lambda^2)}\right]B_0\right\}\cos\tau -$$
$$-\left\{B_0[\alpha_3 + \rho_4(1)] + \tfrac{1}{2}\left[\rho_5(1) - \rho_2(1)\frac{\partial\rho_1}{\partial(\lambda^2)}\right]A_0\right\}\sin\tau + \ldots$$

follows that the necessary condition for the periodicity of σ_3 gives

$$\alpha_3^{(1,2)} = \left\{-\rho_4(1) \pm \tfrac{1}{2}\left[\rho_5(1) - \rho_2(1)\frac{\partial\rho_1}{\partial(\lambda^2)}\right]\right\}_{b=7/12}$$

As the quantity

$$\left[\rho_5(1) - \rho_2(1)\frac{\partial\rho_1}{\partial(\lambda^2)}\right]_{b={}^7/_{12}} = s\left(\mu\frac{2680}{72} + \frac{1948}{72}\right)$$

does not vanish ($\mu > 1$), there exist always two distinct solutions for $\alpha_3[\alpha_3^{(1)} < \alpha_3^{(2)}]$ and the region of instability which we want to find is determined by the following inequalities:

$$1 - \rho_2(1)k^2 + \alpha_3^{(1)}k^3 + \ldots \leqslant \lambda^2 \leqslant 1 - \rho_2(1)k^2 + \alpha_3^{(2)}k^3 + \ldots \quad (3.15)$$

In order to determine the region of instability corresponding to the neighbourhood of $\lambda = 2$ we substitute $\lambda^2 = 4 + \alpha_1 k + \alpha_2 k^2 + \ldots$ in Equation (3.10) and assume the series solution

$$\sigma = A_0\cos 2\tau + B_0\sin 2\tau + k\sigma_1 + k^2\sigma_2 + \ldots$$

with periodic (period π) coefficients (A_0 and B_0 are arbitrary constants). Then, from the equation for the first approximation

$$\frac{d^2\sigma_1}{d\tau^2} + 4\sigma_1 = -A_0\alpha_1\cos 2\tau - B_0\alpha_1\sin 2\tau - \frac{\rho_1(4)}{2}(A_0\sin 4\tau + B_0 - B_0\cos 4\tau)$$

follows, that the condition for periodicity of σ_1 gives $\alpha_1^2 = 0$ and the general solution of this equation is

$$\sigma_1 = A_1\cos 2\tau + B_1\sin 2\tau + \tfrac{1}{24}\rho_1(4)(A_0\sin 4\tau - B_0\cos 4\tau - 3B_0)$$

where A_1 and A_2 are arbitrary constants. For the approximation

$$\frac{d^2\sigma_2}{d\tau^2} + 4\sigma_2 = -A_0\left[\alpha_2 + \rho_2(4) + \tfrac{1}{48}\rho_1^2(4) + \tfrac{1}{2}\rho_3(4)\right]\cos 2\tau -$$
$$- B_0\left[\alpha_2 + \rho_2(4) - \tfrac{5}{48}\rho_1^2(4) - \tfrac{1}{2}\rho_3(4)\right]\sin 2\tau + \ldots$$

the necessary condition for periodicity has the form

$$\left\{\alpha_2 + \rho_2(4) + \tfrac{1}{48}\rho_1^2(4) + \tfrac{1}{2}\rho_3(4)\right\} \times \quad (3.16)$$
$$\times\left\{\alpha_2 + \rho_2(4) + {}_{48}\,\rho_1^2(4) + \tfrac{1}{2}\rho_3(4) - \left[\rho_3(4) + \tfrac{1}{8}\rho_1^2(4)\right]\right\} = 0$$

Since under the condition $s < 1$ the expression

$$8\rho_3(4) + \rho_1^2(4) = 32\,[4s^2(4b^2 - 7b + 3) + 1]$$

does not vanish, it follows that Equation (3.16) determines two distinct values for $\alpha_2[\alpha_2^{(1)} < \alpha_2^{(2)}]$ and the region of instability which we want to find is determined by the following inequalities:

$$4+\alpha_2^{(1)}k^2+\ldots\leqslant\lambda^2\leqslant 4+\alpha_2^{(2)}k^2+\ldots \qquad (3.17)$$

The boundaries of the region of instability in the case $s = 0$ ($n = 1$) could be found from the inequality (3.17). Indeed, in this case we have from Formula (1.4) that $\phi_0 = 1/2\,\pi$, and that the coefficient $F(\tau;\ \lambda^2;\ k)$ is a continuous, periodic function with the period $1/2\,\pi$ and for sufficiently small values of k an analytic function of k^2. Therefore, replacing the independent variable τ in Equation (3.10) by a new variable $\tau_1 = 2\tau$, the period of the function $F(\tau;\ \lambda^2;\ k)$ will be again π and we shall have

$$\frac{d^2\sigma}{d\tau_1^2}+\frac{1}{4}F(\tau_1;\lambda^2;k^2)\,\sigma=0 \qquad (3.18)$$

Since the quantity $1/4\,\lambda^2$ can assume an integral value only in the case when $n = 2$, it follows that after replacing in Formula (3.16) $\rho_i(4)$ by $1/4\,\rho_i(4)$ we obtain two distinct values for α_1

$$\alpha_1^{(1)}=2-4b, \qquad \alpha_2^{(2)}=3-4b$$

and the region of instability is determined by the inequalities

$$1+(2-4b)k^2+\ldots\leqslant\frac{1}{4}\lambda^2\leqslant 1+(3-4b)k^2+\ldots \qquad (3.19)$$

Since the transformation coefficient (3.4) is a periodic continuous function, nonvanishing in the entire period, we can say that the characteristic equation of the system (2.2) has only one unit root; if λ is outside the region of instability, that we have a conjugate pair of complex roots of unit modulus; if λ is inside the region of instability we have two roots, one numerically greater, another numerically smaller than unity. If λ is on the boundary of the region of instability, then the characteristic equation of the system (2.2) has a triple unit root.

In the first case, the solutions of Equation (2.2) will be stable, in the second case unstable. The stability in the third case is of no interest to us.

When $x_0 > y_0$ we could use all the previously-derived formulas after performing in them the following substitution:

$$x_0\leftrightarrow y_0, \quad A\leftrightarrow B, \quad t\leftrightarrow -t, \quad \xi\leftrightarrow\eta, \quad v\leftrightarrow w \qquad (3.20)$$

We shall mention, that in this way we could conduct an analysis when moments of inertia A, B, C have completely arbitrary values, that is, at any $n = 3, 4, 5, \ldots$

4. Thus, when the parameters A, B, C, x_0, y_0, ψ_0 characterizing the distribution of mass in a solid and in the unperturbed motion, respect-

ively, satisfy the condition (3.11) and any one of the conditions (3.13), (3.15), (3.17), (3.19) with the inequality sign, then the characteristic equation of the system (2.2) and (2.3) will have one root numerically greater and one root numerically smaller than unity and the remaining four roots equal to unity. If the above-mentioned parameters satisfy any of the relations (3.13), (3.15), (3.17), (3.19) with the equality sign, then the characteristic equation will have six unit roots; when the parameters A, B, C, x_0, y_0, ψ_0 do not satisfy any of the above mentioned relations, then the characteristic equation will have a conjugate pair of complex roots of unit modulus, and four unit roots.

If all restrictions on initial perturbations are removed, then in the first approximation, the instability will take place with respect to the variables r, γ, γ', and in the case when the parameters A, B, C, x_0, y_0, ψ_0, besides the condition of smallness for ψ_0 and the condition (3.11) satisfy also any of the relations (3.13), (3.15), (3.17), (3.19) with the inequality sign, then the instability will take place with respect to the variables p, q, γ''. In this case, according to the well-known results of Liapunov [3], the unperturbed motion will also be unstable. However, imposing on the initial perturbations certain restrictions we shall have a conditional stability with respect to all variables p, q, r, γ, γ', γ'' in the first approximation, and a conditional stability of the unperturbed motion [3].

In the case when the parameters A, B, C, x_0, y_0, ψ_0 do not satisfy any of the above mentioned relations, then in the first approximation and under the restriction $u(0) = 0$ we shall have the conditional stability with respect to all the variables p, q, r, γ, γ', γ''.

In the case when the parameters A, B, C, x_0, y_0, ψ_0 satisfy any one of the mentioned relations with the equality sign, then we shall have the conditional stability with respect to all the variables p, q, r, γ, γ', γ'' in the first approximation with certain additional restrictions on the initial conditions besides the one $u(0) = 0$. These restrictions would depend on the number of groups of solutions corresponding to the triple unit root of the characteristic equation of the system (2.2).

The stability of the unperturbed motion in the last two cases turns out to be critical and requires an additional investigation.

The performed investigation allows us to estimate the stability of the motion of a solid determined by Formulas (1.2), (1.5) for sufficiently small values of k.

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